

ON TORSION IMAGES OF COXETER GROUPS AND QUESTION OF WIEGOLD

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ABSTRACT. We show that every Coxeter group that is not virtually abelian and for which all labels in the corresponding Coxeter graph are powers of 2 or infinity can be mapped onto uncountably many infinite 2-groups which, in addition, may be chosen to be just-infinite, branch groups of intermediate growth. Also we answer affirmatively a question raised by Wiegold in Kourovka Notebook.

1. INTRODUCTION

This note is a shortened and modified version of the publication [17] (there is a free access to the content of the journal “Algebra and Discrete Mathematics” via <http://adm.lnpu.edu.ua/index.htm>).

One of the most outstanding problems in Algebra known as the Burnside Problem (on periodic groups) was formulated by Burnside in 1902 and was later split into three branches: the General Burnside Problem, the Bounded Burnside Problem, and the Restricted Burnside Problem. The General Burnside Problem was asking if there exists an infinite finitely generated torsion group. It was answered positively by Golod in 1964 [10] based on Golod-Shafarevich Theorem [11]. The Bounded Burnside Problem was solved by S. P. Novikov and S. I. Adjan [35, 1]. The Restricted Burnside Problem was solved by E. Zelmanov [43, 44] as a corollary of his fundamental results on Lie and Jordan algebras. The problem of Burnside inspired a lot of activity and new directions of research. For solution of these problems, various constructions, and surveys we recommend [1, 2, 41, 36, 21, 13, 24, 27, 43, 44, 26, 29, 23, 42, 25, 6, 19, 4, 38, 8, 39] which contain further information on this topic.

Among various problems around the Burnside problem is the problem on minimal values of periods of elements. In the case of the Bounded Burnside Problem the main remaining open question is: what is the minimal n such that the free Burnside group

$$B(m, n) = \langle a_1, \dots, a_m \mid X^n = 1 \rangle$$

given by $m \geq 2$ generators and the identity $X^n = 1$ is infinite? Is it 5, 7, 8 or a larger number? (it is known that the exponents 3, 4, and 6 produce finite free Burnside groups). By the celebrated result of E. Zelmanov [43, 44] finitely generated torsion group with bounded periods of elements cannot be residually finite. Therefore in a finitely generated residually finite torsion group periods of elements are not uniformly bounded and one can study the growth of the period function as was initiated in [13]. For instance, the group \mathcal{G} , constructed by the author in [21] as a

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simple example of a residually finite 2-group, has polynomial growth of periods and is just-infinite (i.e. it is infinite but every proper quotient of it is finite). Therefore making the order of any element of \mathcal{G} smaller will make the group finite.

Fixing the number of generators $m \geq 2$ one may be interested in the minimal values of orders of generators, of products of their powers, of products of length 3 etc, that m -generated infinite residually finite torsion group may have. The case of p -groups is of special interest because of many reasons. For $m = 2$ the order 2 for the generators x, y is impossible because the group would be a dihedral group in this case. As we will see, the orders 2 and 4 (and 8 for the product xy) are possible values, while the triple 2, 4, 4 is not possible (because the corresponding group is crystallographic). Starting with $m = 3$ the orders of generators may take the minimal possible value 2, and we come to the question on torsion quotients of Coxeter groups, which is the main topic of this note. As Coxeter groups are generated by involutions it is natural to investigate their 2-torsion quotients.

Recall that a Coxeter group can be defined as a group with a presentation

$$\mathcal{C} = \langle x_1, x_2, \dots, x_n \mid x_i^2, (x_i x_j)^{m_{ij}}, 1 \leq i < j \leq n \rangle,$$

where $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ (the case $m_{i,j} = \infty$ means that there is no defining relator involving x_i and x_j).

If $m_{i,j} = 2$ this means that x_i and x_j commute. A Coxeter group can be described by a Coxeter graph \mathcal{Z} . The vertices of the graph are labeled by the generators of the group \mathcal{C} , the vertices x_i and x_j are connected by an edge if and only if $m_{i,j} \geq 3$, and an edge is labeled by the corresponding value $m_{i,j}$ whenever this value is 4 or greater. If a Coxeter graph is not connected, then the group \mathcal{C} is a direct product of Coxeter subgroups corresponding to the connected components. Therefore we may focus on the case of connected Coxeter graphs. If we are interested in 2-torsion quotients of \mathcal{C} , then one has to assume that $m_{i,j}$ are powers of 2 or infinity. In order for \mathcal{C} to have infinite torsion quotients it has to be infinite and not virtually abelian. The list of finite and virtually abelian Coxeter groups with connected Coxeter graphs is well known. A comprehensive treatment of Coxeter groups can be found in M. Davis' book [7].

Theorem 1.1. *Let \mathcal{C} be a non virtually abelian Coxeter group defined by a connected Coxeter graph \mathcal{Z} with all edge labels $m_{i,j}$ being powers of 2 or infinity. If \mathcal{Z} is not a tree or is a tree with ≥ 4 vertices, or is a tree with two edges with one label ≥ 4 and the other ≥ 8 , then the group \mathcal{C} has uncountably many 2-torsion quotients. Moreover these quotients can be chosen to be residually finite, just-infinite, branch 2-groups of intermediate growth and the main property that distinguishes them is the growth type of the group.*

Observe that all cases of connected Coxeter graphs that are excluded by the statement of Theorem 1.1 are related to finite or virtually abelian crystallographic groups. Indeed, in the case when \mathcal{Z} consist of one edge the corresponding group is a dihedral group, and when \mathcal{Z} has two edges labeled by 4 the corresponding Coxeter group is the crystallographic group $\langle x, y, z \mid x^2, y^2, z^2, (yz)^2, (xy)^4, (xz)^4 \rangle$ generated by reflections in sides of an isosceles right triangle.

On the other hand, there are four "critical" Coxeter groups Ξ , Φ , Υ , and Π :

$$\Xi = \langle a, c, d \mid a^2, c^2, d^2, (cd)^2, (ad)^4, (ac)^8 \rangle,$$

$$\Phi = \langle x, y, z \mid x^2, y^2, z^2, (xy)^4, (xz)^4, (yz)^4 \rangle,$$

$$\Upsilon = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, (ac)^2, (ad)^2, (bd)^2, (ab)^4, (bc)^4, (cd)^4 \rangle,$$

$$\Pi = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, (bc)^2, (bd)^2, (cd)^2, (ab)^4, (ac)^4, (ad)^4 \rangle,$$

that satisfy the requirements of Theorem 1.1 and play a crucial role in the proof. Their Coxeter graphs are depicted in Figure 1.

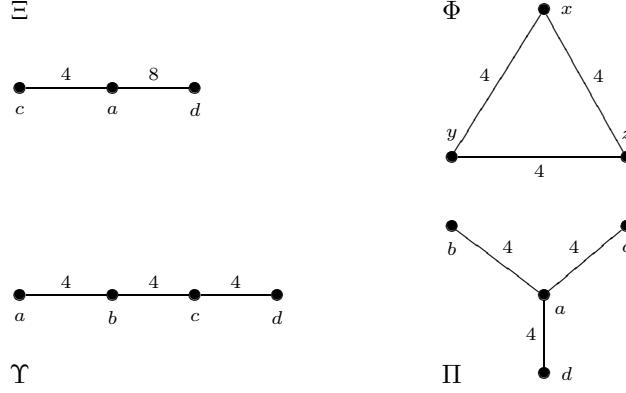


FIGURE 1. Coxeter graphs corresponding to Ξ , Φ , Υ , and Π

The proof of the theorem is based on the properties of the group \mathcal{G} and of the groups of intermediate growth from the uncountable family $\{G_\omega \mid \omega \in \Omega\}$ constructed in [13], which includes (and generalizes) the example \mathcal{G} (some information about groups G_ω will be provided below).

The definition of a branch group is a bit involved and we direct the reader to [16, 15, 4] for more information on branch groups. A group G is a branch group if it has a strictly decreasing sequence $\{H_n\}_{n=0}^\infty$ of normal subgroups of finite index with trivial intersection, satisfying the following properties:

$$[H_{n-1} : H_n] = m_n \in \mathbb{N},$$

for $n = 1, 2, \dots$, there is a decomposition of H_n into the direct product of $N_n = m_1 m_2 \dots m_n$ copies of a group L_n such that the decomposition for H_{n+1} refines the decomposition for H_n (in the sense that each factor of H_n contains the product of m_{n+1} factors of the decomposition of H_{n+1}), and for each n the group G acts transitively by conjugation on the set of factors of H_n . Branch groups constitute one of three classes into which the class of just-infinite groups naturally splits and they appear in various situations [16, 3, 34, 5].

The natural language to work with branch groups is via their actions on regular rooted trees as described in [16, 20, 4]. Then, by definition, a group G acting by automorphisms on a binary rooted tree T (without change the definition holds also for arbitrary spherically homogenous rooted tree) is branch if it acts transitively on levels and for any $n \geq 1$ the rigid stabilizer $\text{rist}_G(n)$ of level n has finite index in G . (Rigid stabilizer of level n is the subgroup generated by the rigid stabilizers of the vertices at level n , and the rigid stabilizer of a vertex u is the subgroup of G acting trivially outside the subtree T_u with root u). Observe that $\text{rist}_G(n)$ is the direct product of $\text{rist}_G(u)$, where u runs over the set of vertices of level n , which makes a link to the algebraic definition given before.

Let $st_G(1)$ be the stabilizer of the first level. Then $\psi : st_G(1) \rightarrow A \times B$ is an embedding, where A and B are the projections of G on the left and right, respectively, rooted subtree of T with roots at the first level.

The groups $G_\omega, \omega \in \Omega_1$ (the sets $\Omega, \Omega_0, \Omega_1$ will be defined later), and in particular the group \mathcal{G} , are branch, just-infinite groups [13, 16] (the term *branch group* is not used in [13] as at the time of writing of the paper there was no definition of this class of groups, but the proof of [13, Theorem 2.2] implies the branch property).

The subgroups $\langle b, ac \rangle, \langle c, ad \rangle, \langle d, ab \rangle$ of index 2 in \mathcal{G} , and the corresponding subgroups $\langle b_\omega, ac_\omega \rangle, \langle c_\omega, ad_\omega \rangle, \langle d_\omega, ab_\omega \rangle$ of index 2 in $G_\omega, \omega \in \Omega_0$ are also branch, because they act transitively on binary tree T as one can easily check or apply the criterion from [16, Theorem 2]. As any proper quotient of a branch group is virtually abelian [16, Theorem 4], and as all groups $G_\omega, \omega \in \Omega_0$ are branch 2-groups, they are just-infinite, as well as are just-infinite the subgroups of index 2 listed above.

A finitely generated group has intermediate growth if the growth function $\gamma(n)$, counting the number of elements of length at most n , grows faster than any polynomial but slower than any exponential function λ^n , for $\lambda > 1$. We use Milnor's equivalence on the set of growth functions of finitely generated groups: $\gamma_1(n) \sim \gamma_2(n)$ if there is $C \in \mathbb{N}$ such that $\gamma_1(n) \leq \gamma_2(Cn)$ and $\gamma_2(n) \leq \gamma_1(Cn)$, for $n = 0, 1, 2, \dots$. For a given finitely generated group the class of equivalence of its growth function does not depend on the choice of a finite generating set and is called the growth degree of the group. It is shown in [13] that there are uncountably many growth degrees of finitely generated groups and, moreover, the partially ordered set of growth degrees of finitely generated groups contains both chains and antichains of continuum cardinality. Some additional information about the growth properties of the family $\{G_\omega\}, \omega \in \Omega_1$ will be provided in the next section.

2. PRELIMINARY FACTS

The group \mathcal{G} was defined in [21] as a group generated by four interval exchange transformations a, b, c, d of order 2 acting on the interval $[0, 1]$ from which the dyadic rational points are removed. From the definition it immediately follows that the generators satisfy the relations

$$a^2 = b^2 = c^2 = d^2 = [b, c] = [b, d] = [c, d] = bcd = (ad)^4 = (ac)^8 = (ab)^{16} = 1$$

(this list of relations is not complete). The branch algorithm for decision of the word problem described in [13] is very efficient and has time (or space) complexity $n \log(n)$. As shown by I. Lysënok [28], \mathcal{G} can be described by the following presentation

$$(2.1) \quad \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \alpha^n((ad)^4), \alpha^n((adacac)^4), n \geq 0 \rangle,$$

where α is the substitution $\alpha : a \rightarrow aca, b \rightarrow d, c \rightarrow b, d \rightarrow c$. It is very interesting and surprising that the relators in Lysënok presentations are words of power at most 8, as we know nothing about the free Burnside group of exponent 8. The group \mathcal{G} is not finitely presented, and it is shown in [14] that the relators given in (2.1) are independent (i.e., none of them can be deleted from the set of relators without changing the group). The relation $bcd = 1$ implies that the group \mathcal{G} is 3-generated, but it is usually convenient to work with the generating set $\mathcal{A} = \{a, b, c, d\}$, because together with the identity element it constitutes the so called nucleus of the group,

an important tool in the study of self-similar groups [34]. Excluding the generator b we see that \mathcal{G} is a homomorphic image of the group Ξ .

For the proof of Theorem 1.1 we will use the construction of an uncountable family of groups G_ω , where $\omega \in \Omega = \{0, 1, 2\}^\mathbb{N}$ described in [13] for which the group \mathcal{G} is a particular case corresponding to the sequence $\varsigma = (012)^\infty$. The group G_ω is generated by the set of elements $\mathcal{A}_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ of order 2, with $b_\omega, c_\omega, d_\omega$ commuting and generating the Klein 4-group (i.e. $b_\omega c_\omega d_\omega = 1$) (so indeed the groups G_ω are 3-generated). For the definition of these groups we address the reader to [13, 18]. Originally G_ω were defined similarly to \mathcal{G} as groups acting on $[0, 1]$ (with removed diadic rational points), but more convenient language to work with them is via action on binary sequences (via identification of a point from $[0, 1]$ with its binary expansion), or via actions by automorphisms on a binary rooted tree T , when we identify vertices of the tree with corresponding binary sequences.

Let Q be a subgroup of Ξ generated by the elements $x = a, y = d, z = cac$. It is easy to check that Q has index 2 in Ξ and has a presentation

$$\langle x, y, z \mid x^2, y^2, z^2, (xy)^4, (xz)^4, (yz)^4 \rangle.$$

Therefore Q is isomorphic to Φ .

Let $\Omega_0 \subset \Omega$ be the subset consisting of sequences ω which contain each symbol 0, 1, 2 infinitely many times, $\Omega_1 \subset \Omega$ be the set of sequences which contain at least two symbols from $\{0, 1, 2\}$ infinitely many times, and $\Omega_2 = \Omega \setminus \Omega_1$ be the set of sequences $\omega = \omega_1 \omega_2 \dots \omega_n \dots$ such that $\omega_n = \omega_{n+1} = \omega_{n+2} = \dots$ starting with some coordinate n . Observe that all sets $\Omega_0, \Omega_1, \Omega_2$ are invariant with respect to the shift τ

$$\tau(\omega_1 \omega_2 \omega_3 \dots) = \omega_2 \omega_3 \dots$$

in the space of sequences. The groups G_ω are virtually abelian for $\omega \in \Omega_2$, while the groups G_ω , for $\omega \in \Omega_1$ are just-infinite, branch groups of intermediate growth. Additionally, the groups G_ω , for $\omega \in \Omega_0$ are 2-groups. Proofs of these facts are provided by Theorems 2.1, 2.2, 8.1, and Corollary 3.2 in [13]. One of important facts that will be used in the proof of the Theorem 1.1 is that the set of growth degrees of groups $G_\omega, \omega \in \Omega_0$ has uncountable cardinality. The word problem for the family $G_\omega, \omega \in \Omega_1$ can be solved by algorithm with oracle ω (i.e. the algorithm which uses the symbols of the sequence ω in its work), which we call branch algorithm because of its branching nature [18, 13]. Using this algorithm, or directly from the definition of groups G_ω , it is easy to check that if ω begins with symbol 0 then $(ad_\omega)^4 = 1$, if $\omega = 1 \dots$, then $(ac_\omega)^4 = 1$ and if $\omega = 2 \dots$, then $(ab_\omega)^4 = 1$. As we can exclude any of $b_\omega, c_\omega, d_\omega$ from the generating set we see that each of the groups $G_\omega, \omega \in \Omega_1$ is a homomorphic image of Ξ . To simplify the situation we assume that ω begins with 0, so $(ad_\omega)^4 = 1$. Let $\Omega_3 \subset \Omega_0$ be the set of sequences which begin with symbol 0. The proofs of results about growth in [13] allow to conclude that the set of growth degrees of groups from $\{G_\omega, \omega \in \Omega_3\}$ has uncountable cardinality. Moreover the same holds for any set of the form $w\Omega_0$, where w is arbitrary finite binary sequence.

The results from [13] also show that the group $G_\omega, \omega \in \Omega_1$ is abstractly commensurable with $G_{\tau(\omega)} \times G_{\tau(\omega)}$ and therefore the growth of G_ω is equal to the square of the growth of $G_{\tau(\omega)}$.

3. PROOF OF THE THEOREM

Proof. First we show that a Coxeter group \mathcal{C} , satisfying the condition of the theorem 1.1 can be mapped onto one of Coxeter groups Ξ , Φ , Υ , or Π . This will reduce the proof to these groups. Indeed everything will be deduced from the fact that the group Ξ satisfies the conclusion of the theorem.

Assume that the graph \mathcal{Z} is not a tree, so it contains a cycle of length ≥ 3 consisting of vertices $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ for some $3 \leq k \leq n$. Taking the quotient of \mathcal{C} by the normal subgroup generated by the generators x_j which do not belong to this cycle, we can pass to the case when the graph \mathcal{Z} is a cycle. Taking the quotient by the relation $x_{i_1} = x_{i_2}$ (if the length of the cycle is greater than 3) we make the cycle shorter. After finitely many steps of this type we come to the case when the length of the cycle is 3. Then making the further factorization by replacing the numbers $m_{i,j} \geq 8$ by $m_{i,j} = 4$, we map \mathcal{C} onto Φ .

If \mathcal{Z} is a tree, passing to an appropriate quotient reduces the situation to the case when the graph \mathcal{Z} looks like a “segment” (all vertices are of degree ≤ 2) with 3 or 4 vertices, and labeling of edges given by the set $\{4, 8\}$ or $\{4, 4, 4\}$ respectively, or like a tripod “Y” (i.e. is a tree with four vertices, one of degree 3 and three leaves) with all edges labeled by 4, which correspond to the cases of groups Ξ , Υ and Π respectively.

We already know from the previous section that Ξ has uncountably many quotients $G_\omega, \omega \in \Omega_3$, with different types of growth which are branch just-infinite 2-groups. Φ is a subgroup of index 2 in Ξ . Let Q_ω be the corresponding quotient of Φ in G_ω of index 2. Obviously Q_ω is 2-group and has the same growth type as G_ω . The group Q_ω (as well as G_ω) acts on binary rooted tree T , and for branch property we need only to show that the action is level transitive because for each n the rigid stabilizer $\text{rist}_{Q_\omega}(n)$ has index ≤ 2 in $\text{rist}_{G_\omega}(n)$. But Q_ω acts transitively on the first level and both projections of $st_{Q_\omega}(1)$ are equal to the subgroup $R_\omega = \langle d_{\tau(\omega)}, ac_{\tau(\omega)} \rangle$ which is of index 2 in $G_{\tau(\omega)}$. This subgroup also acts transitively on the first level and both projections of $st_{R_\omega}(1)$ are equal to the group $G_{\tau^2(\omega)}$, which is branch. Therefore R_ω and Q_ω act transitively by [16, Theorem 4] and are branch groups. We conclude that Φ has uncountably many quotients satisfying conclusion of theorem 1.1.

Now we are going to consider the case of Υ . Let Λ be a subgroup of index 2 in Φ generated by the elements $u = xy, v = xz$. Then Λ has a presentation

$$\Lambda = \langle u, v \mid u^4, v^4, (uv)^4 \rangle.$$

Consider the subgroup \mathcal{S} of \mathcal{G} generated by the elements ad and $(ac)^2$. It is a quotient of Λ with respect to the map

$$u \rightarrow ad, \quad v \rightarrow (ac)^2.$$

Computations show that $\psi(st_{\mathcal{S}}(1))$ is a subgroup in $\mathcal{G} \times \mathcal{G}$ generated by the pairs $(b, b), (da, ad), (bac, da), (badac, (da)^2)$, and the projections of this subgroup on each factor is the group $\langle b, ac \rangle = \langle b, ad \rangle$, which has index 2 in \mathcal{G} and is branch. Therefore \mathcal{S} acts transitively on levels, is branch, and just-infinite.

Let S_ω , for $\omega \in \Omega_3$ be the subgroups of G_ω generated by ad_ω and $(ac_\omega)^2$. Then the relators of Λ are also relators of S_ω with respect to the map

$$\nu : x \rightarrow ad_\omega, \quad v \rightarrow (ac_\omega)^2.$$

The image $\psi(st_{S_\omega}(1))$ is a branch subgroup $\langle b_\omega, ac_\omega \rangle$ of index 2 in G_ω . Therefore S_ω is also branch, just-infinite and has the same growth type as G_ω .

Let $\bar{\Lambda}$ be any of the 2-quotients S_ω of Λ given by the previous arguments, and let u, v be the set of generators of $\bar{\Lambda}$ which are the images of the generators of Λ (we keep the same notation for them). Consider the group $\bar{\Lambda}_1$, acting on binary rooted tree T , generated by the element a of order two (permutation of two subtrees T_0, T_1 with roots at the first level) and the elements $b = (1, v), c = (u, u), d = (v, v)$, where u, v and the identity element act on the left or right subtree respectively (in a same way they act on the whole tree; here we use the self-similarity of the binary tree). Then a commutes with c and d , b commutes with d , and $(ab)^4 = (bc)^4 = (cd)^4 = 1$, so the group is a quotient of Υ . The ψ -image of stabilizer of the first level of $\bar{\Lambda}_1$ is a subdirect product of $\bar{\Lambda} \times \bar{\Lambda}$ and contains the group $D \times D$ where D is the normal closure of v in $\bar{\Lambda}$. As $\bar{\Lambda}$ is just-infinite, D has finite index in $\bar{\Lambda}$. Therefore the growth of $\bar{\Lambda}_1$ is equal to the square of the growth of $\bar{\Lambda}$. It is clear that $\bar{\Lambda}_1$ is branch and just-infinite. As the set of squares of growth degrees $\bar{\Lambda}$ has uncountable cardinality, we are done with this case.

Now consider the last case of the group Π . Let $G = G_\omega, \omega \in \Omega_3$ be a 2-group, whose generators will be denoted, for simplicity, by a, b, c, d instead of $a, b_\omega, c_\omega, d_\omega$. Recall that a acts by permutation of the two subtrees T_0, T_1 of the binary tree with roots on the first level. Consider the group $V = \langle a, \bar{a}, \bar{b}, \bar{c} \rangle$, where $\bar{a}, \bar{b}, \bar{c}$ are automorphisms of the tree fixing the vertices of the first level whose ψ -images are $(a, 1), (1, b), (1, c)$ respectively (here again we use the self-similarity of binary rooted tree identifying T with T_0, T_1). Then the generators $a, \bar{a}, \bar{b}, \bar{c}$ are of order 2, $\bar{a}, \bar{b}, \bar{c}$ commute, and $(a\bar{a})^4 = (a\bar{b})^4 = (a\bar{c})^4 = 1$, so the group is a homomorphic image of the Π with respect to the map

$$a \mapsto a, \quad b \mapsto \bar{b}, \quad c \mapsto \bar{c}, \quad d \mapsto \bar{a}.$$

The ψ -image of $st_V(1)$ is a subdirect product of $G \times G$ and contains $A \times A$, where A is the normal closure of a in G (A has finite index in G , as G is just-infinite). V acts transitively on levels and therefore is branch and just-infinite. The growth of V is the square of the growth of G_ω . Therefore Π has uncountably many quotients satisfying the statement of the theorem. \square

4. CONCLUDING REMARKS

In 2006, J. Wiegold raised the following question in Kourovka Notebook [31, 16.101]. Do there exist uncountably many infinite 2-groups that are quotients of the group

$$\Delta = \langle x, y \mid x^2, y^4, (xy)^8 \rangle?$$

The problem is motivated by the following comment by J. Wiegold “There certainly exists one, namely the subgroup of finite index in Grigorchuk’s first group generated by b and ad ; see (R. I. Grigorchuk, *Functional Anal. Appl.*, **14** (1980), 41–43).”

Immediately after the appearance we informed one of the Editors of Kourovka Notebook, I. Khukhro, that the answer to the question is positive, and that the results of [13] can be easily used to provide a justification. Unfortunately, it took some time for the author to write the corresponding text, and he is finally presenting his arguments in this note. Different argument has been used recently in the article [32] and the authors were notified of the approach given here (they acknowledgment this fact at the end of Section 2).

Let L be a subgroup of Ξ generated by $x_1 = ac, x_2 = ad$. Then L is a subgroup of index 2 in Ξ , has a presentation

$$L = \langle x_1, x_2 \mid x_1^4, x_2^8, (x_2 x_1^{-1})^2 \rangle,$$

and therefore is isomorphic to the group Δ via the map $x \rightarrow x_1^{-1}x_2, y \rightarrow x_1$. Let L_ω be the subgroup of G_ω of index 2 generated by ac_ω and ad_ω . Then, if ω begins with 0 (and so $(ad_\omega)^4 = 1$), the group L_ω is a homomorphic image of Δ . As the set of growth degrees of groups $L_\omega, \omega \in \Omega_0, \omega = 0w_2 \dots$ has cardinality 2^{\aleph_0} we get the affirmative answer to the Wiegold question. Obviously L_ω are 2-groups. One can show that they are branch and just-infinite as it is shown in [17]. Observe that alternatively the groups L_ω can be defined as groups generated by elements $x = b_\omega, y = ad_\omega$ as was suggested by Wiegold in the case of \mathcal{G} , and that L_ω satisfy the relations

$$1 = x^2 = y^4 = (xy)^8 = (xy^2)^{16}$$

(the provided list of defining relations is not complete). It is unclear if the power 16 in the last relation can be replaced by 8, i.e. if there is an infinite 2-generated 2-group the set of defining relations of which starts with

$$1 = x^2 = y^4 = (xy)^8 = (xy^2)^8.$$

There are other approaches for construction of infinite torsion quotients of Coxeter groups. For instance, for those Coxeter groups which can be mapped onto non-elementary hyperbolic groups (in Gromov sense [22]), or which are “large” groups in the sense of S. Pride [40, 9] (a group is “large” if it has a subgroup of finite index that can be mapped onto a free group of rank 2), the results and constructions from [25, 37, 8, 32] can be used.

The criterion for a Coxeter group defined by a connected Coxeter graph to be non-elementary hyperbolic, given by G. Moussong in [33], requires that each Coxeter subgroup generated by a subset $\{x_i, x_j, x_k\}$ of three generators is a hyperbolic triangular group, i.e. a group isomorphic to the group $T_{m,n,q}^* = \langle x, y, z \mid x^2, y^2, z^2, (xy)^m = (xz)^n = (yz)^q \rangle$ with

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{q} < 1.$$

The groups Ξ and Φ are non-elementary hyperbolic and, as was indicated by T. Januszkiewicz, the groups Υ and Π can be mapped onto non-elementary hyperbolic groups. Therefore all these groups have uncountably many homomorphic torsion images of bounded degree according to [32].

Indeed, all Coxeter groups which are not virtually abelian are “large”, which is a particular case of the results by G. Margulis and E. Vinberg from [30]. This fact was also proved independently by C. Gonciulea, as is indicated in the A. Lubotzki’s review [MR1748082 (2001h:22016)] to [30], but published only in a weaker form [12]. Therefore in view of the results from [37, 32], for any prime number p and any Coxeter group \mathcal{C} that is not virtually abelian, there is 2^{\aleph_0} pairwise non isomorphic quotients of \mathcal{C} which are residually finite virtually p -groups. It is pointed out by T. Januszkiewicz that it is possible that every Coxeter group that is not virtually abelian has a non-elementary hyperbolic quotient (perhaps this is a known fact). If this is the case, then every Coxeter group that is not virtually abelian has uncountably many torsion quotients of bounded exponent.

Finally let us formulate an open question. The p -groups ($p \geq 3$ is a prime) of Gupta-Sidki [24] are 2-generated, residually finite, branch, and just-infinite. Their generators x, y satisfy the relations $x^p = y^p = (x^i y^j)^{p^2} = 1, 1 \leq i, j \leq p-1$.

Problem 1. *Let $p \geq 5$ be a prime. Does there exist a residually finite p -group generated by two elements x, y subject to the relations $x^p = y^p = (x^i y^j)^p = 1, 1 \leq i, j \leq p-1$? Can such a group have additionally some other finiteness properties (for instance in the spirit of theorem 1.1)?*

The prime $p = 3$ is excluded for obvious reasons.

Observe that the quotient \mathcal{G} of Ξ is a self-similar group (historically it is the first example of a non elementary self-similar group; more on self-similar groups see in [3, 34]). The groups $Q = Q_\zeta, \bar{A}_1, V$ used in the proof of theorem 1.1, which are quotients of groups Φ, Υ and Π respectively, are not self-similar. It would be interesting to find self-similar torsion quotients of Φ, Υ and Π if they exist (or to show that there is no such quotients).

REFERENCES

- [1] S. I. Adian, *The Burnside problem and identities in groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 95, Springer-Verlag, Berlin, 1979. MR 80d:20035
- [2] S. V. Alešin, *Finite automata and the Burnside problem for periodic groups*, Mat. Zametki **11** (1972), 319–328. MR MR0301107 (46 #265)
- [3] Laurent Bartholdi, Rostislav Grigorchuk, and Volodymyr Nekrashevych, *From fractal groups to fractal sets*, Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, pp. 25–118. MR MR2091700
- [4] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šuník, *Branch groups*, Handbook of algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 989–1112. MR MR2035113
- [5] Laurent Bartholdi and Volodymyr V. Nekrashevych, *Iterated monodromy groups of quadratic polynomials. I*, Groups Geom. Dyn. **2** (2008), no. 3, 309–336.
- [6] Laurent Bartholdi and Zoran Šuník, *On the word and period growth of some groups of tree automorphisms*, Comm. Algebra **29** (2001), no. 11, 4923–4964. MR MR1856923 (2002i:20040)
- [7] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR MR2360474 (2008k:20091)
- [8] Thomas Delzant and Misha Gromov, *Courbure mésoscopique et théorie de la toute petite simplification*, J. Topol. **1** (2008), no. 4, 804–836. MR MR2461856
- [9] M. Edjvet and Stephen J. Pride, *The concept of “largeness” in group theory. II*, Groups—Korea 1983 (Kyoungju, 1983), Lecture Notes in Math., vol. 1098, Springer, Berlin, 1984, pp. 29–54. MR MR781355 (86g:20039)
- [10] E. S. Golod, *On nil-algebras and finitely approximable p -groups*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276. MR MR0161878 (28 #5082)
- [11] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272. MR MR0161852 (28 #5056)
- [12] Constantin Goniculea, *Infinite Coxeter groups virtually surject onto \mathbf{Z}* , Comment. Math. Helv. **72** (1997), no. 2, 257–265. MR MR1470091 (98d:20045)
- [13] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR MR764305 (86h:20041)
- [14] ———, *On the system of defining relations and the Schur multiplier of periodic groups generated by finite automata*, Groups St. Andrews 1997 in Bath, I, London Math. Soc. Lecture Note Ser., vol. 260, Cambridge Univ. Press, Cambridge, 1999, pp. 290–317.
- [15] ———, *Branch groups*, Mat. Zametki **67** (2000), no. 6, 852–858. MR MR1820639 (2001i:20057)
- [16] ———, *Just infinite branch groups*, New horizons in pro- p groups, Progr. Math., vol. 184, Birkhäuser Boston, Boston, MA, 2000, pp. 121–179. MR MR1765119 (2002f:20044)

- [17] R.I. Grigorchuk, *On a question of Wiegold and torsion images of Coxeter groups*, Algebra Discrete Math. (2009), no. 4, to appear.
- [18] Rostislav Grigorchuk, *Solved and unsolved problems around one group*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 117–218. MR MR2195454
- [19] Rostislav Grigorchuk and Igor Lysionok, *Burnside problem*, The concise handbook of algebra (Alexander V. Mikhalev and Günter F. Pilz, eds.), Kluwer Academic Publishers, Dordrecht, 2002, pp. 111–115.
- [20] Rostislav Grigorchuk and Zoran Šunić, *Self-similarity and branching in group theory*, Groups St. Andrews 2005, I, London Math. Soc. Lecture Note Ser., vol. 339, Cambridge Univ. Press, Cambridge, 2007, pp. 36–95.
- [21] R. I. Grigorčuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54. MR MR565099 (81m:20045)
- [22] M. Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR MR919829 (89e:20070)
- [23] Narain Gupta, *On groups in which every element has finite order*, Amer. Math. Monthly **96** (1989), no. 4, 297–308. MR MR992077 (90d:20073)
- [24] Narain Gupta and Saïd Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), no. 3, 385–388. MR MR696534 (84g:20075)
- [25] S. V. Ivanov and A. Yu. Ol'shanskii, *On finite and locally finite subgroups of free Burnside groups of large even exponents*, J. Algebra **195** (1997), no. 1, 241–284. MR MR1468892 (98h:20070)
- [26] Sergei V. Ivanov, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. **4** (1994), no. 1-2, ii+308. MR MR1283947 (95h:20051)
- [27] A. I. Kostrikin, *Around Burnside*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 20, Springer-Verlag, Berlin, 1990, Translated from the Russian and with a preface by James Wiegold. MR MR1075416 (91i:20038)
- [28] I. G. Lysënok, *A set of defining relations for the Grigorchuk group*, Mat. Zametki **38** (1985), no. 4, 503–516, 634. MR MR819415 (87g:20062)
- [29] ———, *Infinite Burnside groups of even period*, Izv. Ross. Akad. Nauk Ser. Mat. **60** (1996), no. 3, 3–224. MR MR1405529 (97j:20037)
- [30] G. A. Margulis and È. B. Vinberg, *Some linear groups virtually having a free quotient*, J. Lie Theory **10** (2000), no. 1, 171–180. MR MR1748082 (2001h:22016)
- [31] V. D. Mazurov and E. I. Khukhro (eds.), *The Kourovka notebook*, sixteenth ed., Russian Academy of Sciences Siberian Division Institute of Mathematics, Novosibirsk, 2006, Unsolved problems in group theory, Including archive of solved problems. MR MR2263886 (2007g:20002)
- [32] Ashot Minasyan, Alexander Yu. Olshanskii, and Dmitriy Sonkin, *Periodic quotients of hyperbolic and large groups*, Groups Geom. Dyn. **3** (2009), no. 3, 423–452. MR MR2516174
- [33] Gabor Moussong, *Hyperbolic coxeter groups*, Ph.D. thesis, Ohio State University, 1988.
- [34] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR MR2162164
- [35] P. S. Novikov and S. I. Adjan, *Infinite periodic groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 251–524. MR MR0240179 (39 #1532b)
- [36] A. Ju. Ol'shanskii, *An infinite simple torsion-free Noetherian group*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 6, 1328–1393. MR MR567039 (81i:20033)
- [37] A. Ol'shanskii and D. Osin, *Large groups and their periodic quotients*, 2006, (available at <http://arxiv.org/abs/0601589>).
- [38] A. Yu. Olshanskii and D. V. Osin, *Large groups and their periodic quotients*, Proc. Amer. Math. Soc. **136** (2008), no. 3, 753–759 (electronic). MR MR2361846 (2008j:20115)
- [39] Denis Osin, *Rank gradient and torsion groups*, (available at <http://arxiv.org/abs/0905.1322>), 2009.
- [40] Stephen Pride, *On groups of finite height*, J. Austral. Math. Soc. Ser. A **28** (1979), no. 1, 87–99. MR MR541170 (80m:20028)
- [41] V. I. Sushchansky, *Periodic permutation p -groups and the unrestricted Burnside problem*, DAN SSSR. **247** (1979), no. 3, 557–562, (in Russian).

- [42] Michael Vaughan-Lee, *The restricted Burnside problem*, London Mathematical Society Monographs. New Series, vol. 5, The Clarendon Press Oxford University Press, New York, 1990, Oxford Science Publications. MR MR1057610 (92c:20001)
- [43] E. I. Zel'manov, *Solution of the restricted Burnside problem for groups of odd exponent*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 42–59, 221. MR MR1044047 (91i:20037)
- [44] ———, *Solution of the restricted Burnside problem for 2-groups*, Mat. Sb. **182** (1991), no. 4, 568–592. MR MR1119009 (93a:20063)

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